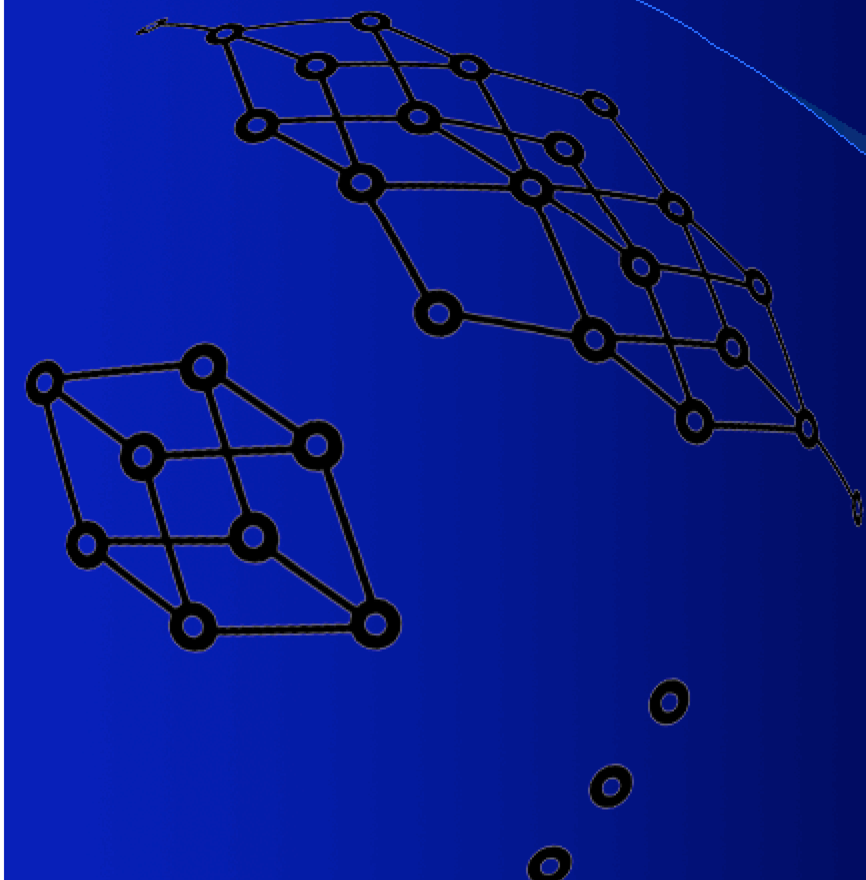
Questions Revisited



Kevin H. Knuth, Ph.D.
Computational Sciences Division
NASA Ames Research Center
Moffett Field CA 94035

“The important thing is not to stop questioning."

-Albert Einstein

Outline

The Space of Questions

The Real Question Sublattice

The Geometry of Questions

Valuations on Questions

Cox's Generalized Entropy

The Space of Questions

Richard Threlkeld Cox 1898-1991

In Cox's last paper, he for the first time looked at the logic of inquiry and the relationships between questions.

Defining a Question

Richard T. Cox (1979) defined a **system of assertions** as a set of assertions, which includes every assertion implying an assertion of the set. A **question** is a system.

The **irreducible set of a system** is a subset which contains every assertion that implies no other assertion than itself.

A **defining set of a system** is a subset which includes the irreducible set.

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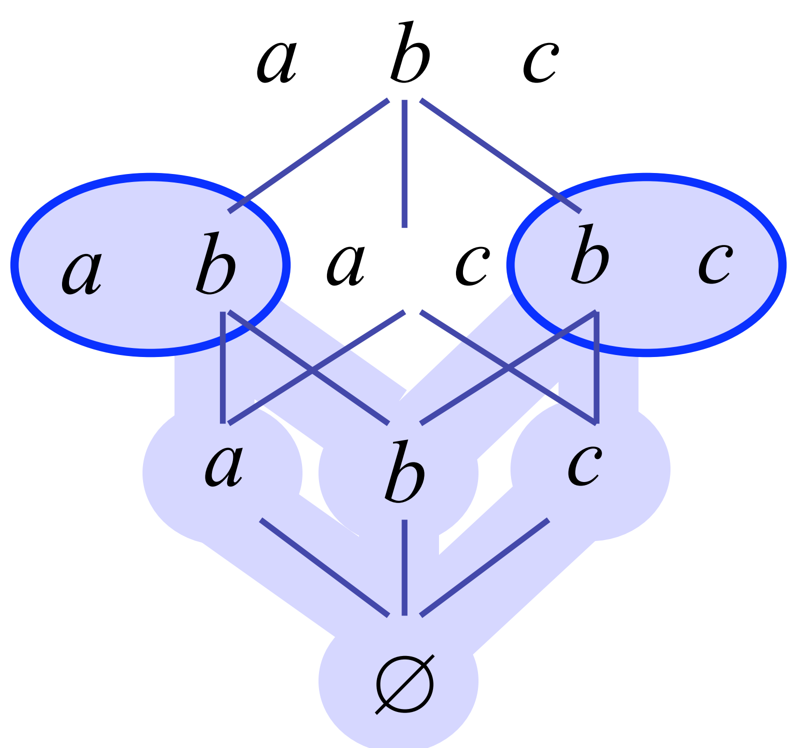


When are Questions Equal?

Two questions are **equivalent** when they are defined by the same system of assertions.

Consider the questions

"Is it raining?"

"Is it not raining?"

They are both answered by the system of assertions described by the irreducible set { "It is raining!", "It is not raining!"} and are therefore equivalent.

Who Stole the Tarts?

T = "Who stole the tarts made by the Queen of Hearts?"

I contrive a simple defining set for T, which I claim is an exhaustive, mutually exclusive, irreducible set.

T = { a = "Alice stole the tarts!",
k = "The Knave of Hearts stole the tarts!",
n = "No one stole the tarts!" }

Real questions can always be answered by a true statement.
Vain questions cannot be answered by any true statement.

Some Questions Answer Others

Now consider the binary question

B = “Did or did not Alice steal the tarts?”

B = {a = “Alice stole the tarts!”, ~a = “Alice did not steal the tarts!”}

As the defining set of T is exhaustive, $\sim a = k \quad n$

Since A is a system of assertions, it must contain all the assertions defining any assertion in the set and therefore contains the set T. Thus

$$T \;\square\; B$$

Ordering Questions

T = “Who stole the tarts made by the Queen of Hearts?”

B = “Did or did not Alice steal the tarts?”

$$T \;\square\; B$$

T* answers *B

B* includes *T

Meets and Joins of Questions

With "is a subset of" as the ordering relation among questions, one can show that

The meet of two questions, the **joint question**, is the set intersection of the set of assertions answering the question.

$$A \wedge B \equiv A \cap B$$

The join of two questions, the **common question**, is the set union of the set of assertions answering the question.

$$A \vee B \equiv A \cup B$$

Duality Again

Cox implicitly defined his ordering relation to be "contains as a subset" which is dual to "is a subset of".

This is the reason that Cox's version of the consistency relations are upside down with respect to the consistency relations for assertions.

I will deviate from Cox and maintain consistency with the conventions of lattice theory and define for questions

$$A \square B \equiv A \square B \equiv A \square B$$

Read either as "A answers B" or "B includes A"

Consistency Relations

Consistency Relations

When $A \,\square\, B$

C1. $A \,\square\, B = A$ (A is the greatest lower bound of A and B)

C2 . $A \quad B = B$ (B is the least lower bound of A and B)

Jointly “Who stole the tarts?” and “Did or did not Alice steal the tarts?” ask “”Who stole the tarts?”

Whereas they ask “Did or did not Alice steal the tarts?” in common.

Ideals

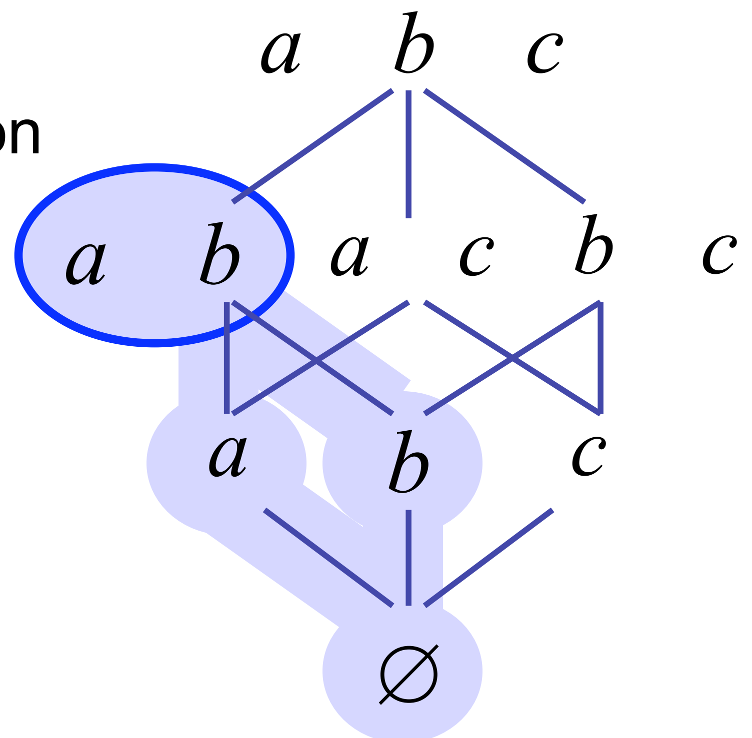
An **ideal** is a nonvoid subset J of a lattice A with the properties (Birkhoff 1967)

I1. $a \square J,\ x \square a$ where $x \square a$ then $x \square J$

I2. $a \square J,\ b \square J$ then $a \quad b \square J$

Note that property I1. Is the condition for the set J to be a system of assertions, or equivalently a question.

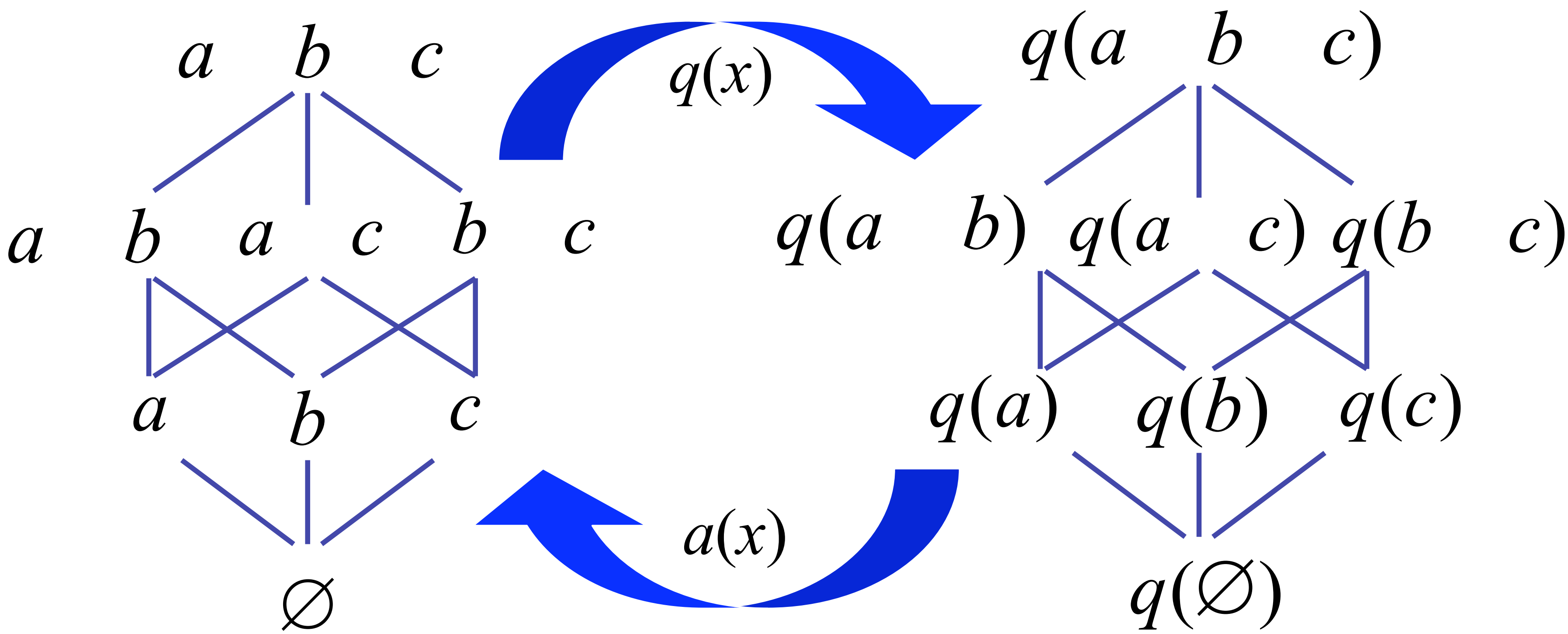
Therefore, as ideals are questions, I call them **Ideal Questions**.



Ideals and Ideal Questions

Given any assertion x in the lattice, I can construct the set $q(x)$ of all assertions y such that y □ x .

Thus $q(x)$ takes an assertion to a question - ISOMORPHIC.



Reflection Symmetry

There are More Questions!

However, I can create new questions by forming joins of the ideal questions.

Here is one $q(a \quad b) \quad q(c) \equiv q(a \quad b) \square q(c)$

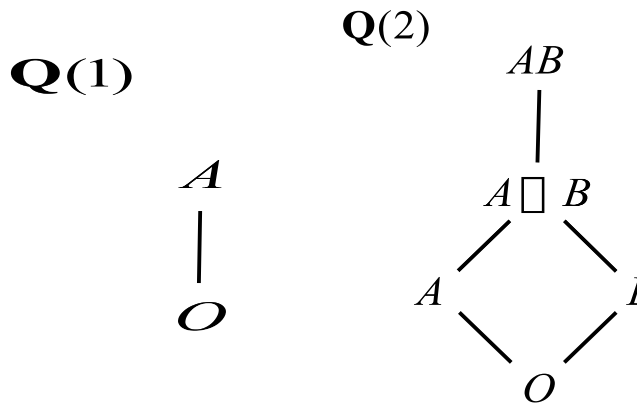
which as a system of assertions is written

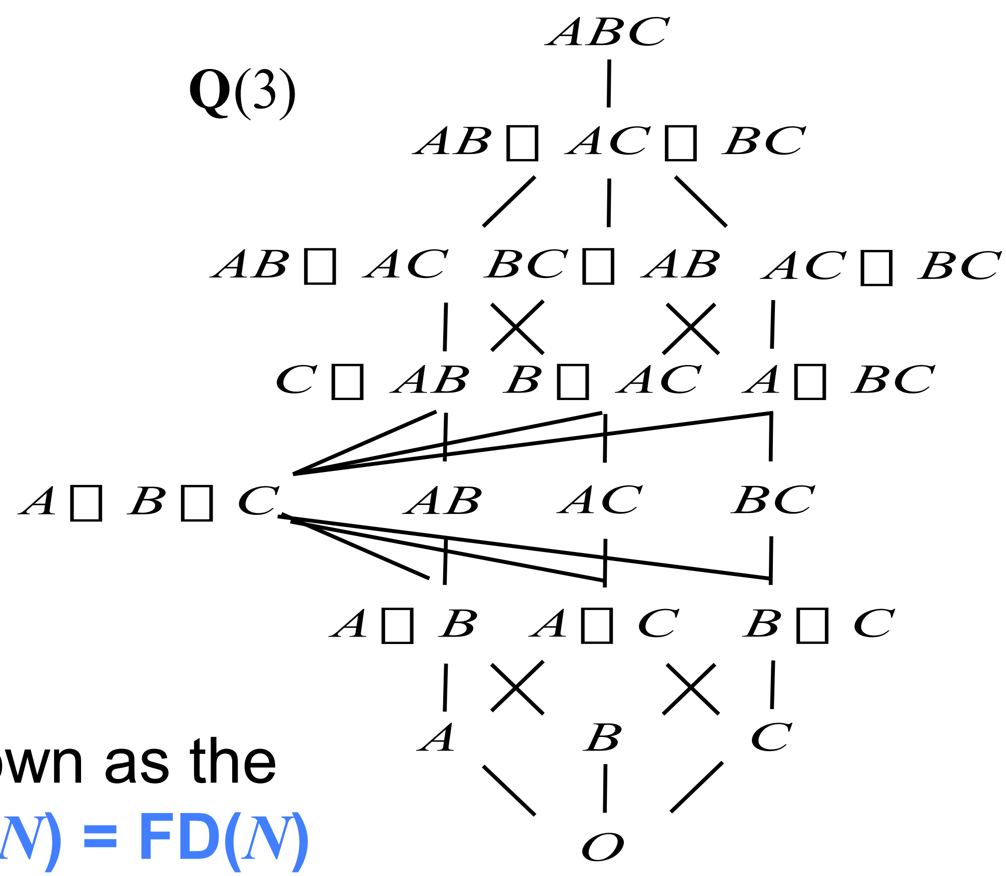
$$\{a \quad b, a, b, \varnothing\} \square \{c, \varnothing\} = \{a \quad b, a, b, c, \varnothing\}$$

I will write them in shorthand as

$$\boxed{AB \square C \equiv q(a \quad b) \square q(c)}$$

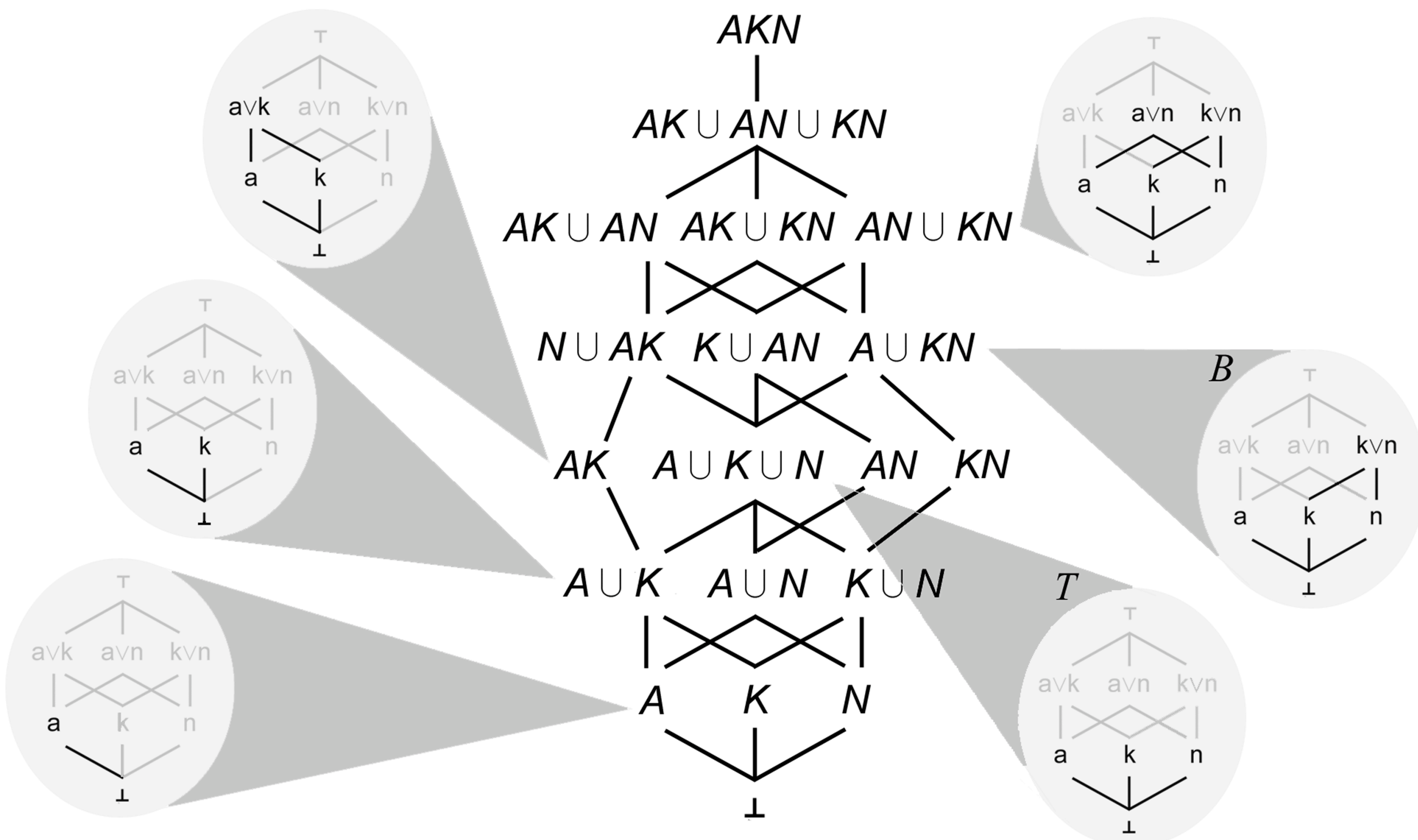
The Lattice of Questions





The lattice of questions is known as the
Free Distributive Lattice, Q(*N*) = FD(*N*)

Who Stole the Tarts?



More Questions Than Answers

The number of assertions goes as 2^N where N is the number of mutually exclusive atoms in the assertion lattice.

The number of elements in a free distributive lattice is an unsolved problem and is known as **Dedekind's Problem**.

Here are the number of corresponding questions given N mutually exclusive atomic assertions for N = 1 through 8:

2, 5, 19, 167, 7 580, 7 828 353, 2 414 682 040 997,

56 130 437 228 687 557 907 787 (Sloane A014466)

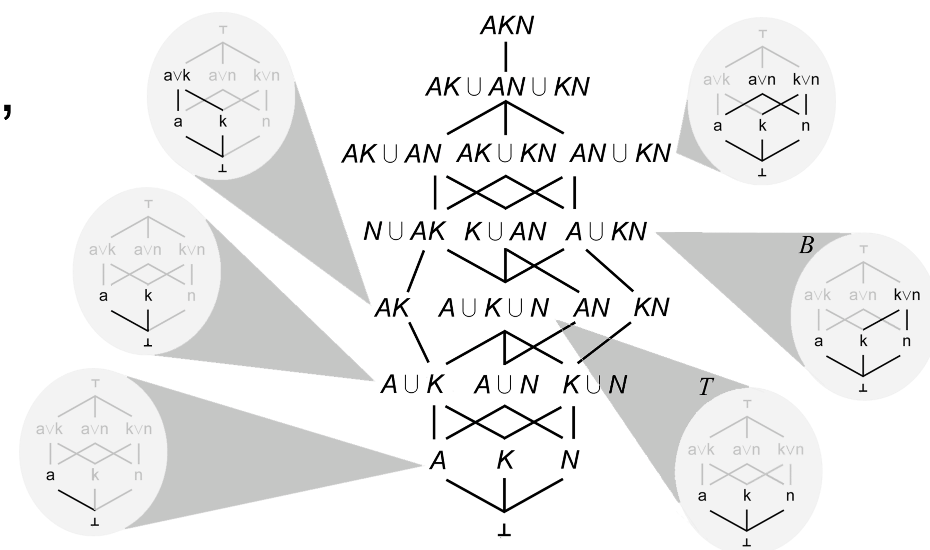
No reflection symmetry between assertions and questions
(Cant form a one-to-one map!)

No Complements to Questions

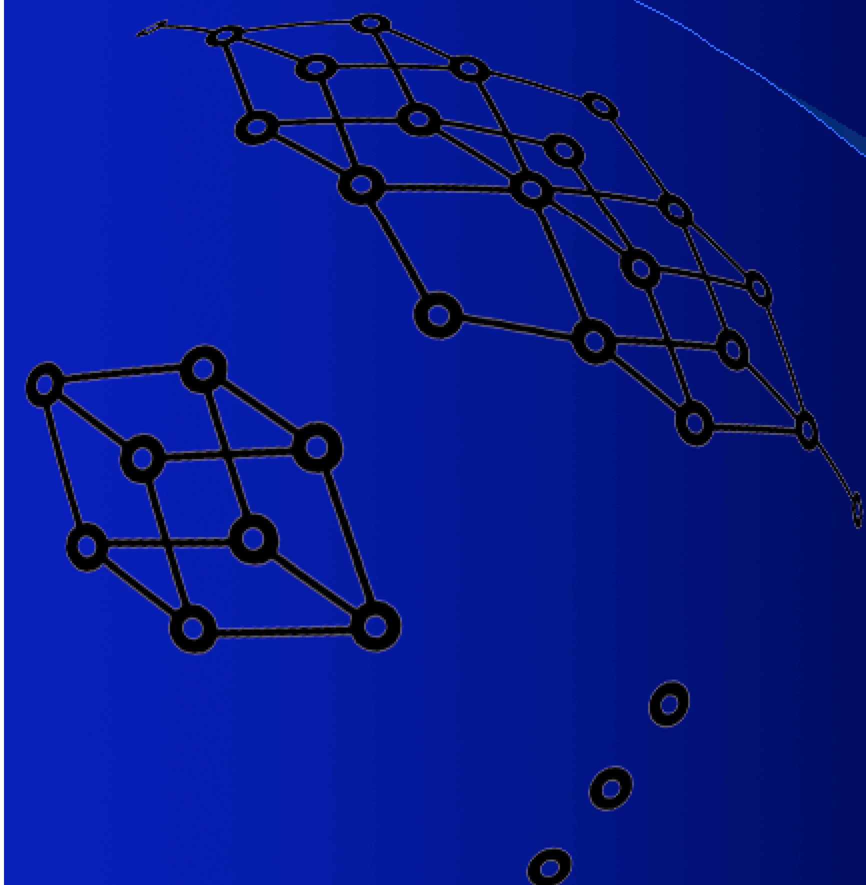
Also, there can be no complements to questions in general.

The join-irreducible elements of $Q(N)$ are the ideal questions, which are isomorphic to the Boolean lattice of assertions.

This is not an antichain. Therefore, the question lattice is not Boolean and is thus not complemented!



Real Question Sublattice

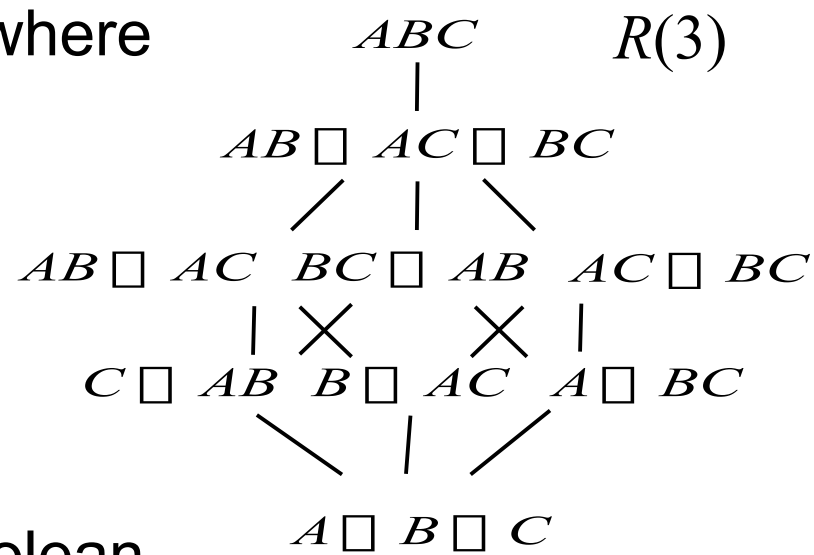


Real Questions

But what about the **Real Questions**?
They are always answered by a true assertion and thus include the exhaustive set of mutually exclusive assertions.

In Q(3) these are the questions R where

$$A \square B \square C \;\; \square \;\; R$$



Does this sublattice have complements?

Except for the top element, it is Boolean.

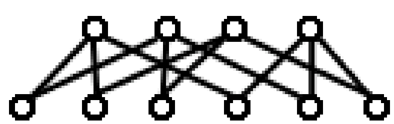
Again No Complements

No, in general there are no complements.

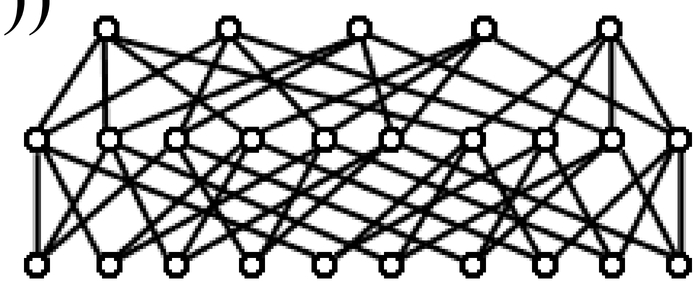
The join irreducible elements of $R(N)$ are the elements that can be written as $\bigvee_{i=1}^{M} q(a_{b_i})$ $\quad q(\bigvee_{j=M+1}^{N} a_{b_j})$

$J(R(3))$

$J(R(4))$



$J(R(5))$



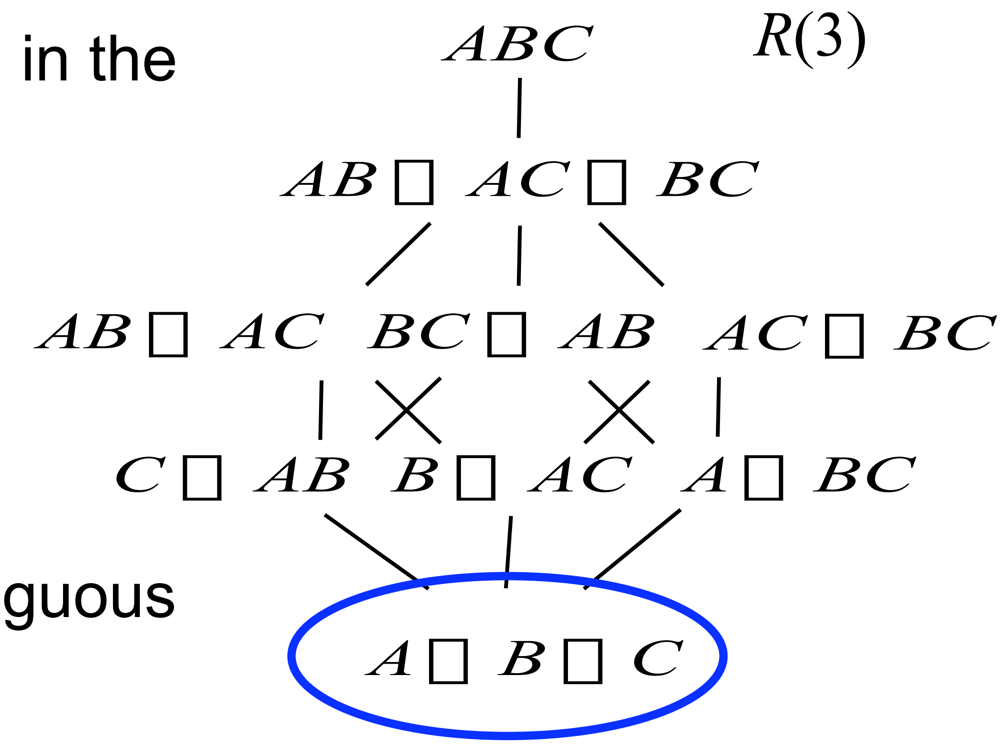
These posets are antichains for $R(1)$, $R(2)$, and $R(3)$, but not for $N > 3$. So in general the algebra of $R(N)$ is not Boolean.

Deductive Inquiry

Cox was right to be interested in the ordering relation
“contains as a subset”
as the question

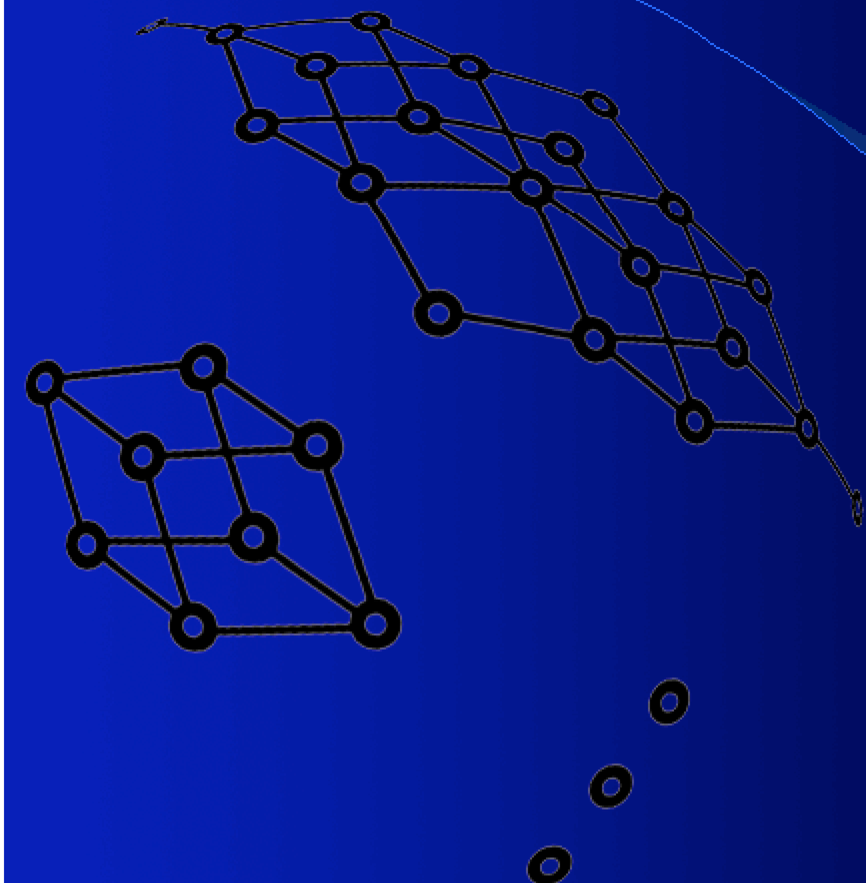
$$A \vee B \vee C$$

is most useful in the sense that it is answered by no ambiguous assertions.

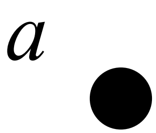


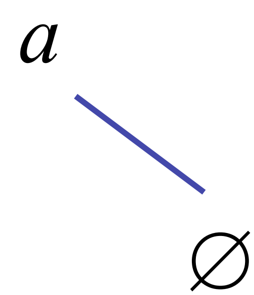
Whichever ordering relation is used, one can define a function analogous to that in the Boolean lattices that describes inclusion.

A Geometric View of Questions



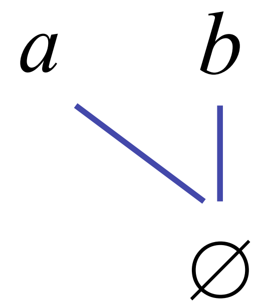
The 0-Simplex



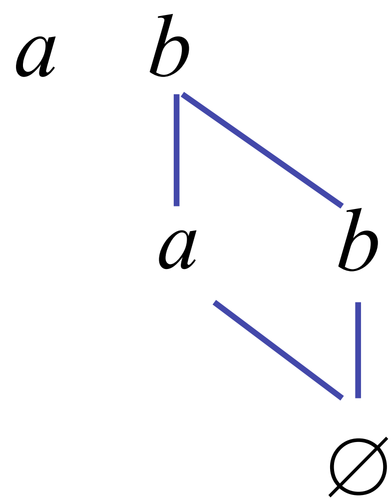


0-simplex

Consider Two 0-Simplexes

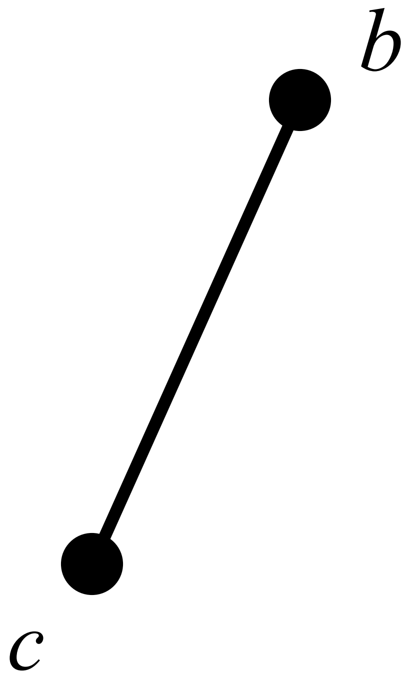


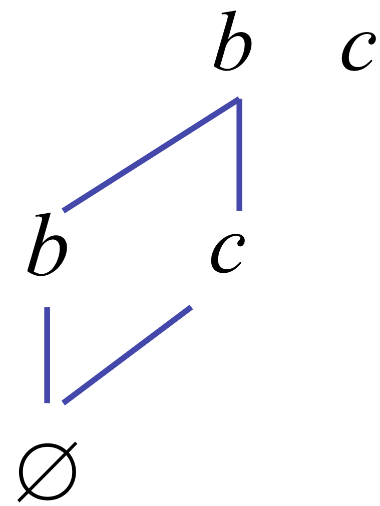
The 1-Simplex



1-simplex

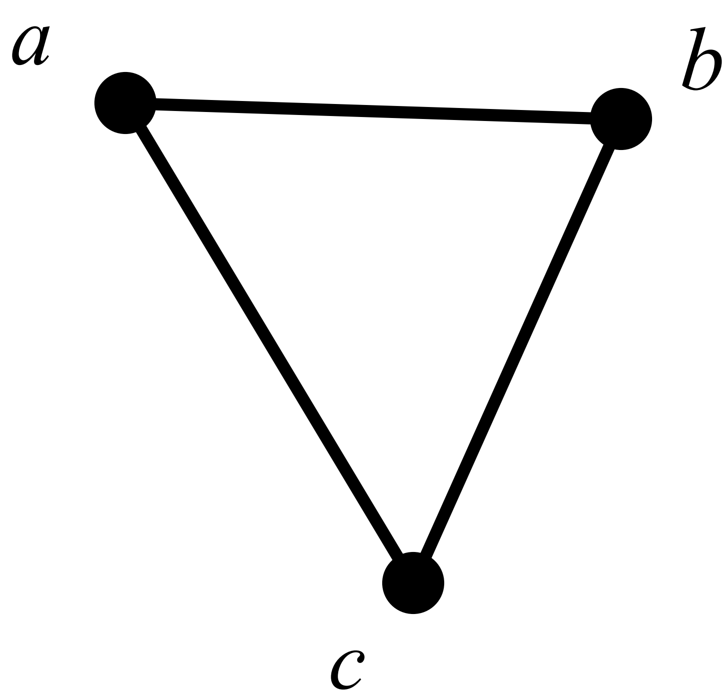
Another 1-Simplex

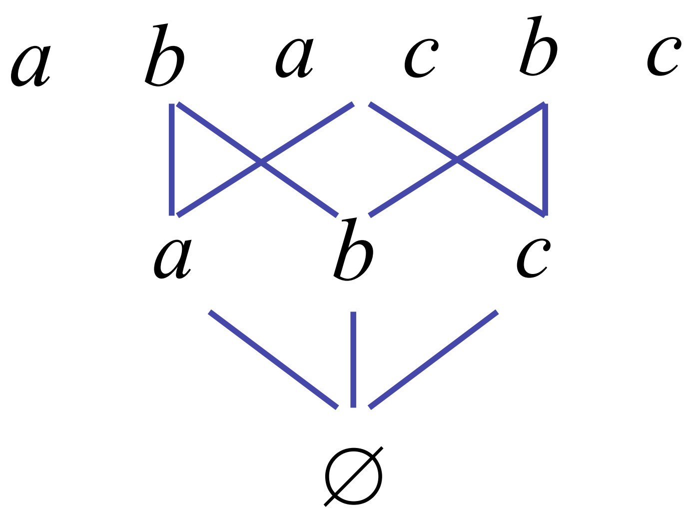




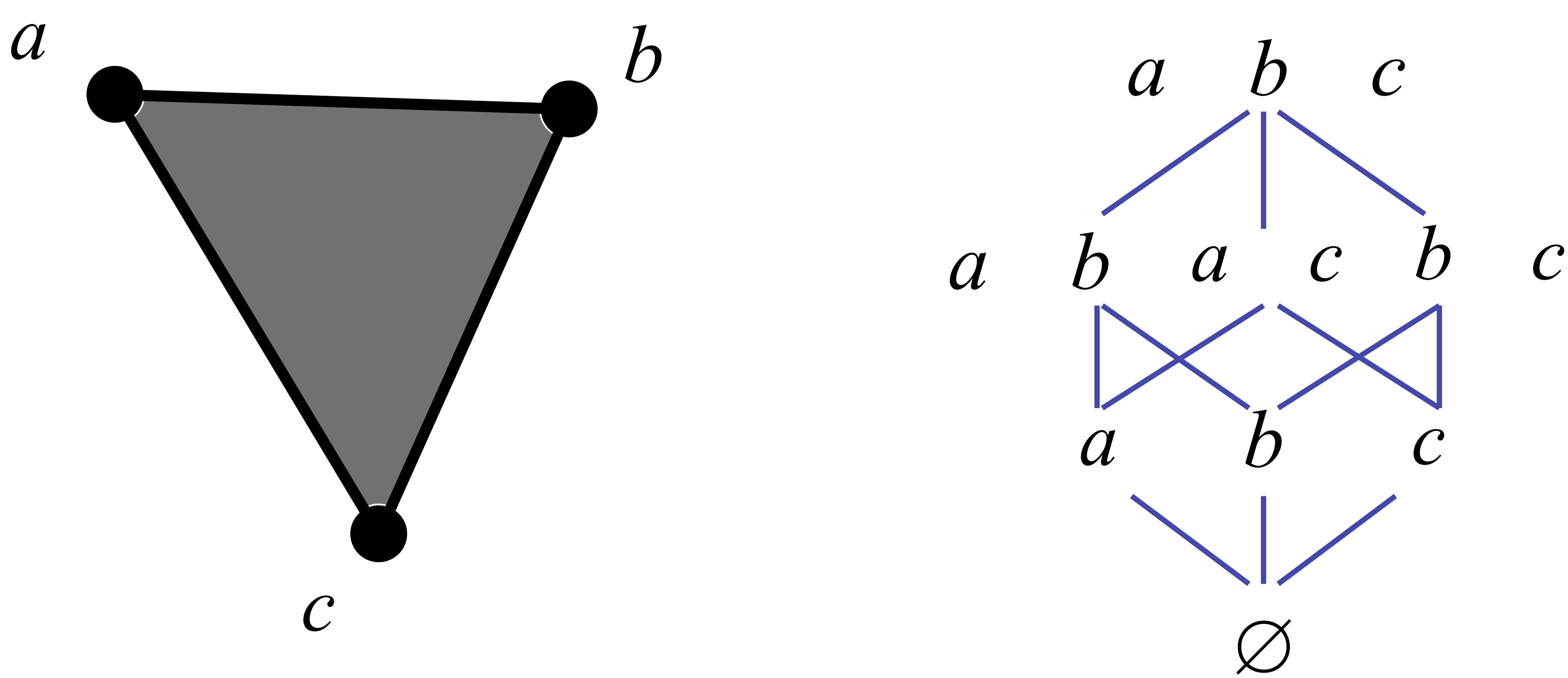
1-simplex

Several 1-Simplexes



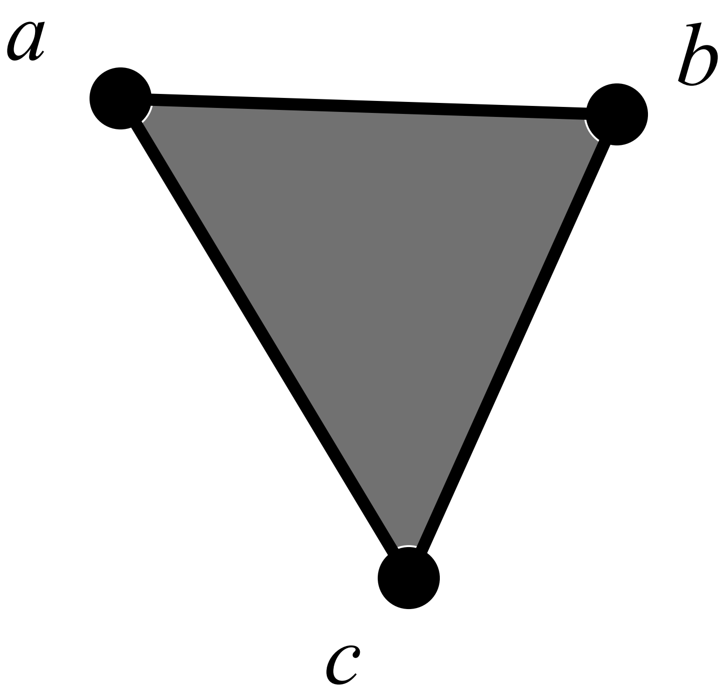


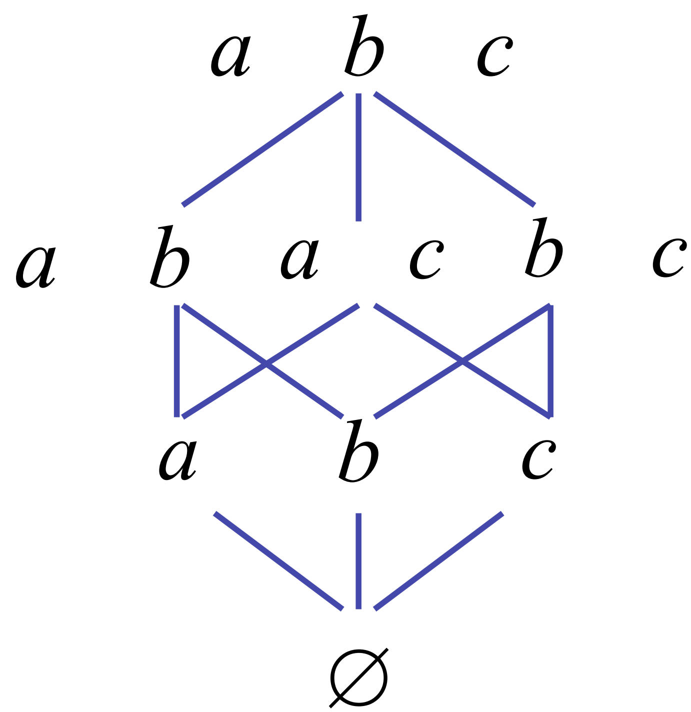
The 2-Simplex



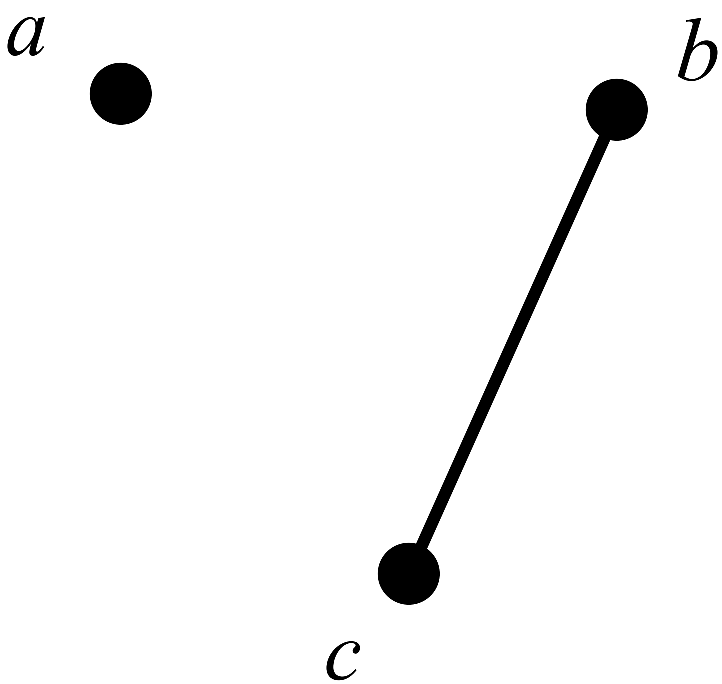
2-simplex

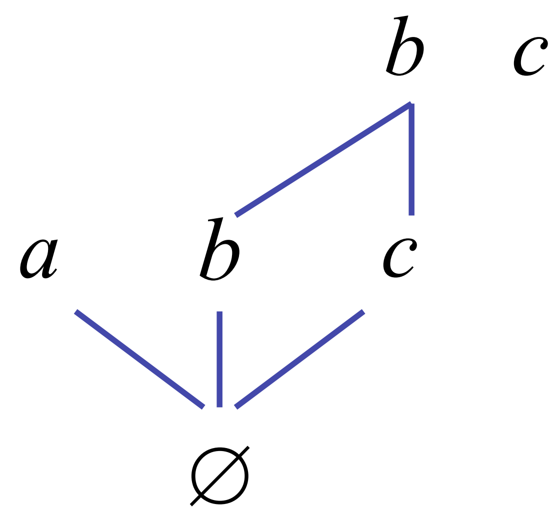
Simplex Structure is Boolean





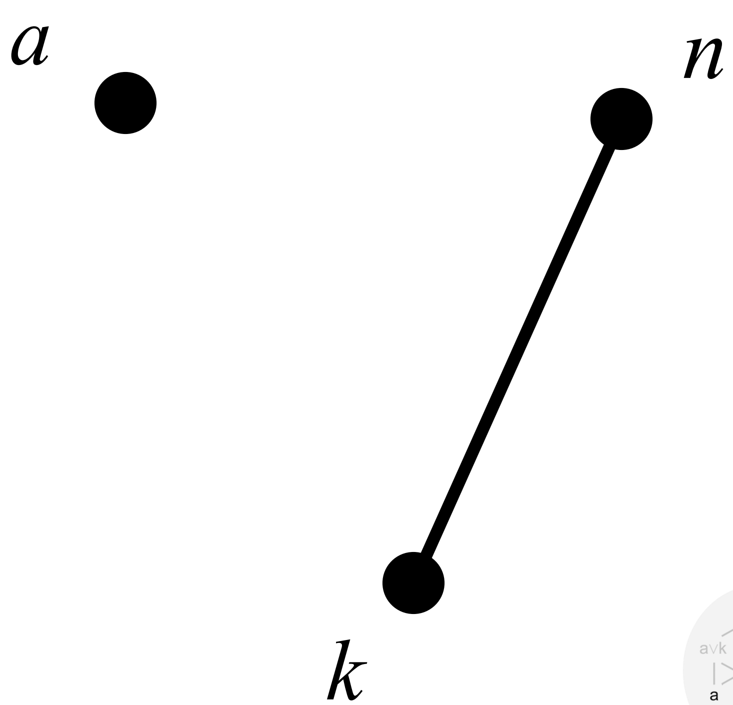
Simplicial Complexes

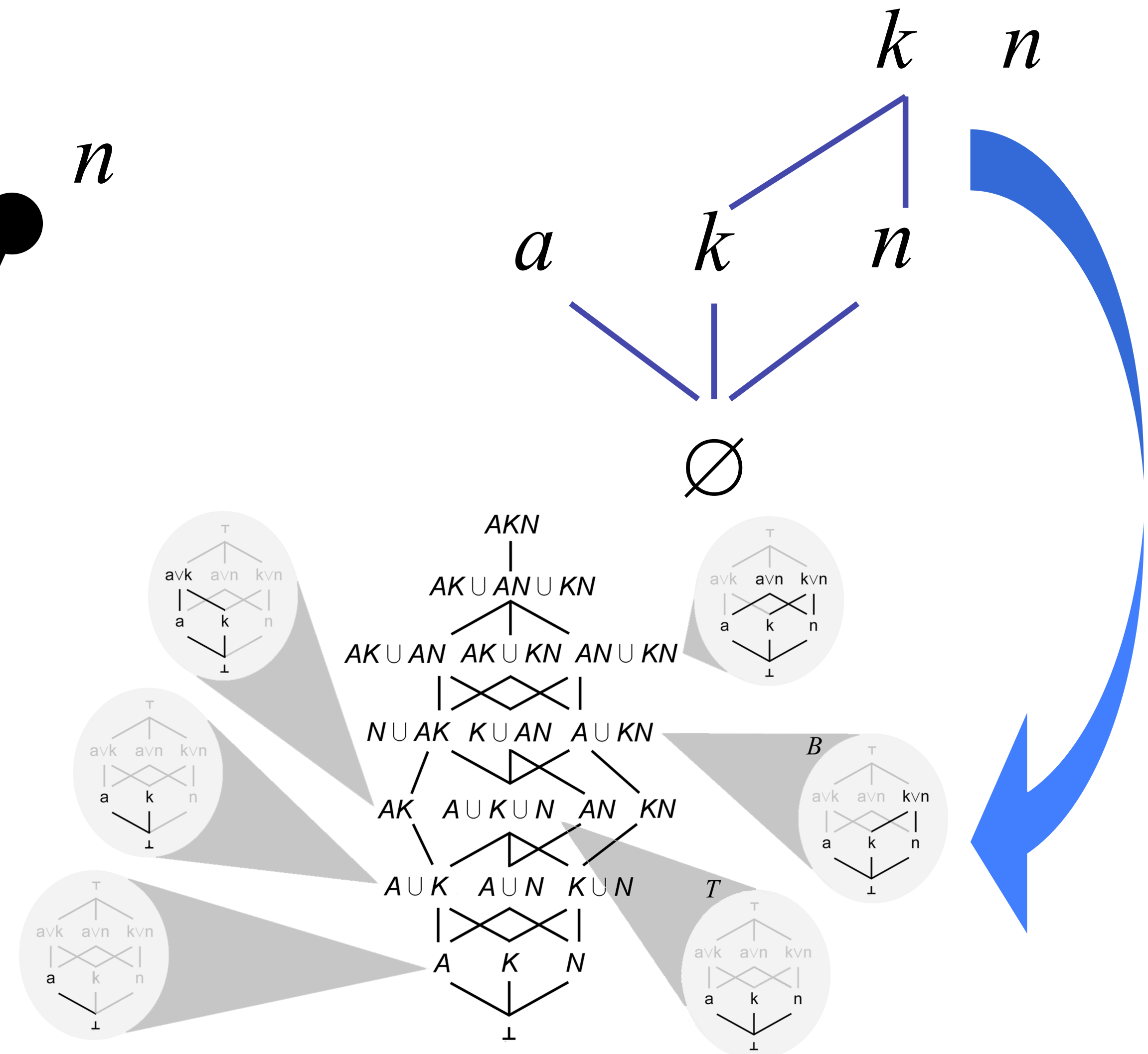




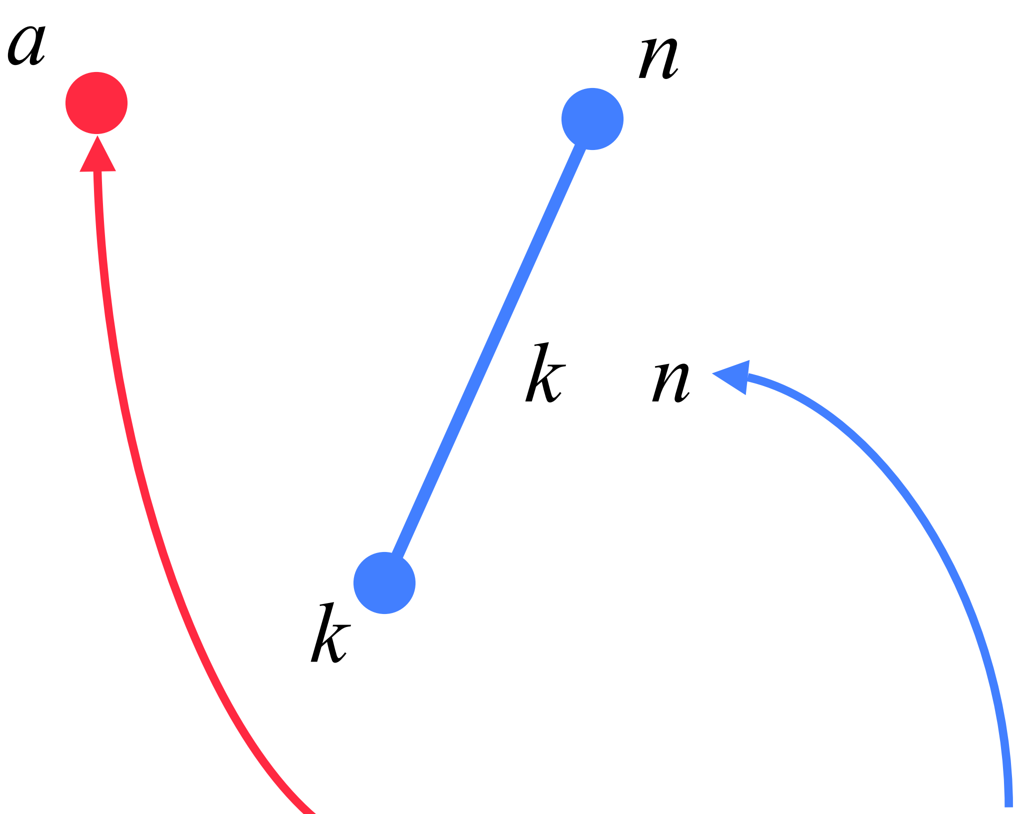
A join of simplexes is called a simplicial complex

Examine this Object



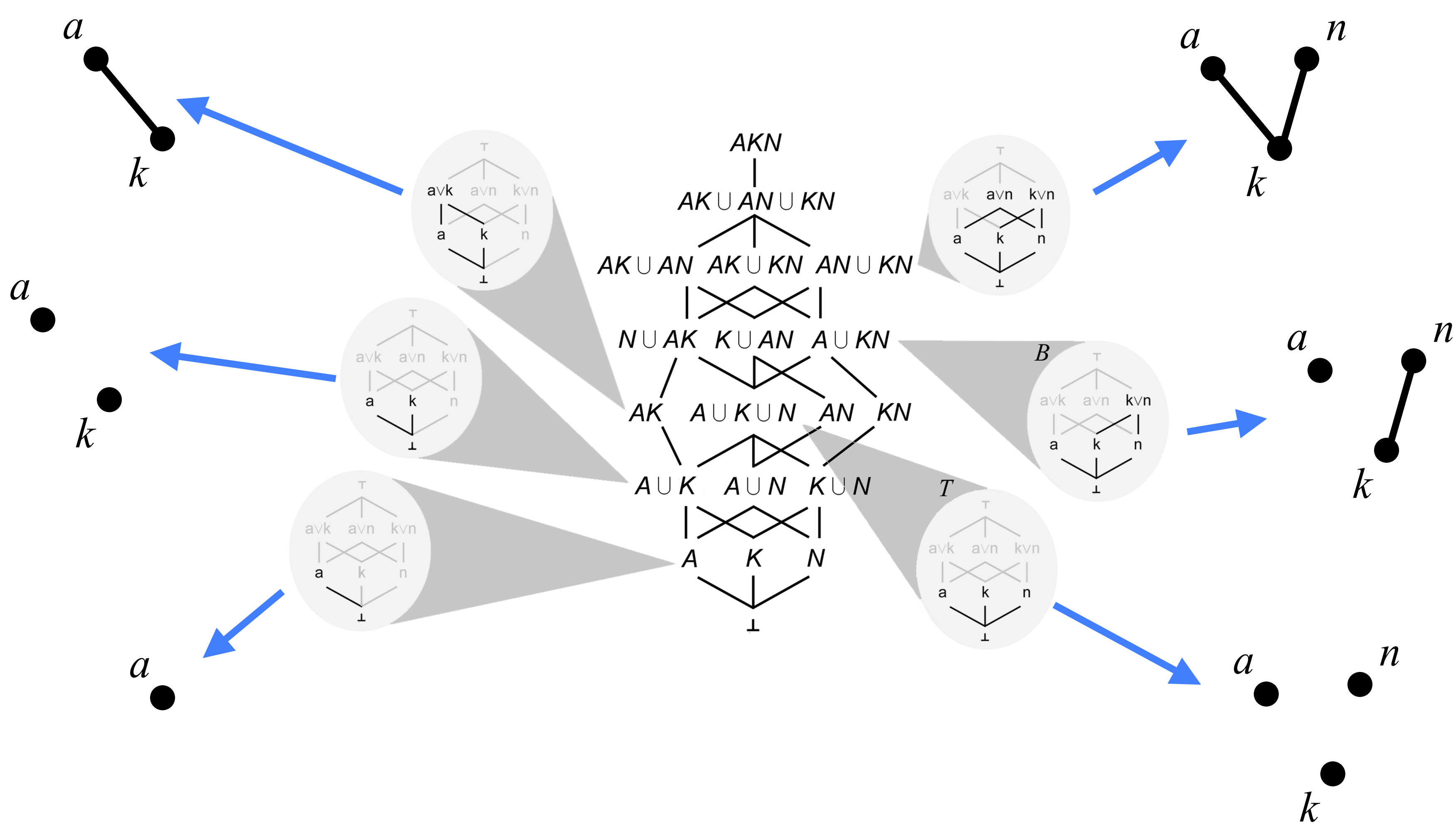


Questions = Simplicial Complexes



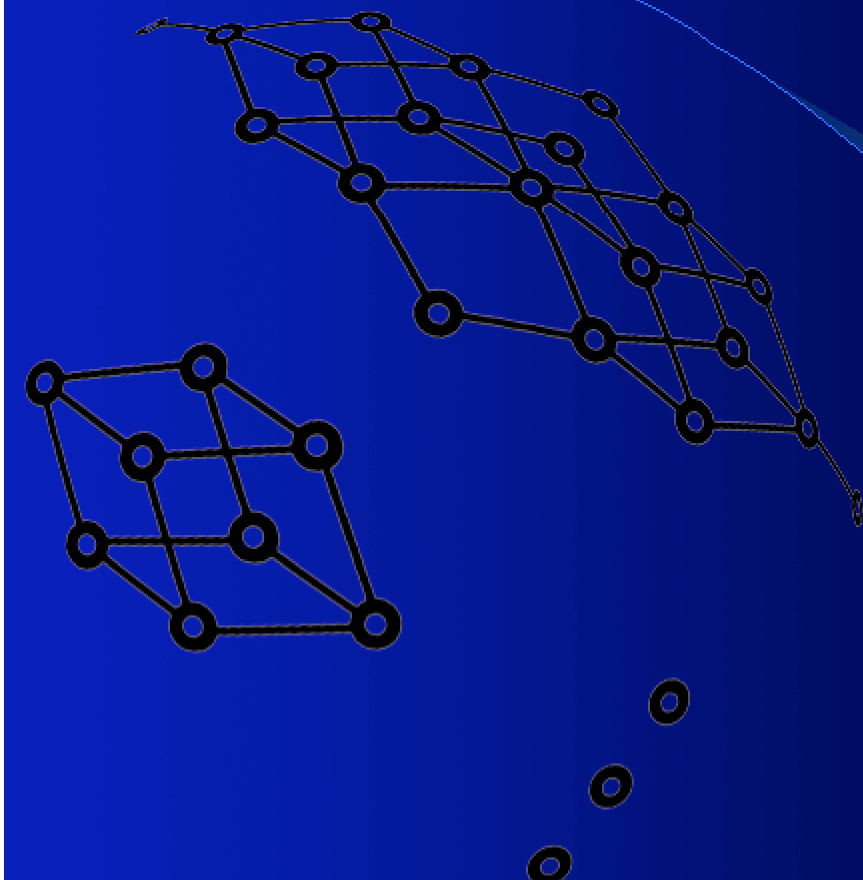
Did Alice or Did Not Alice steal the tarts?

Lattice of Simplicial Complexes

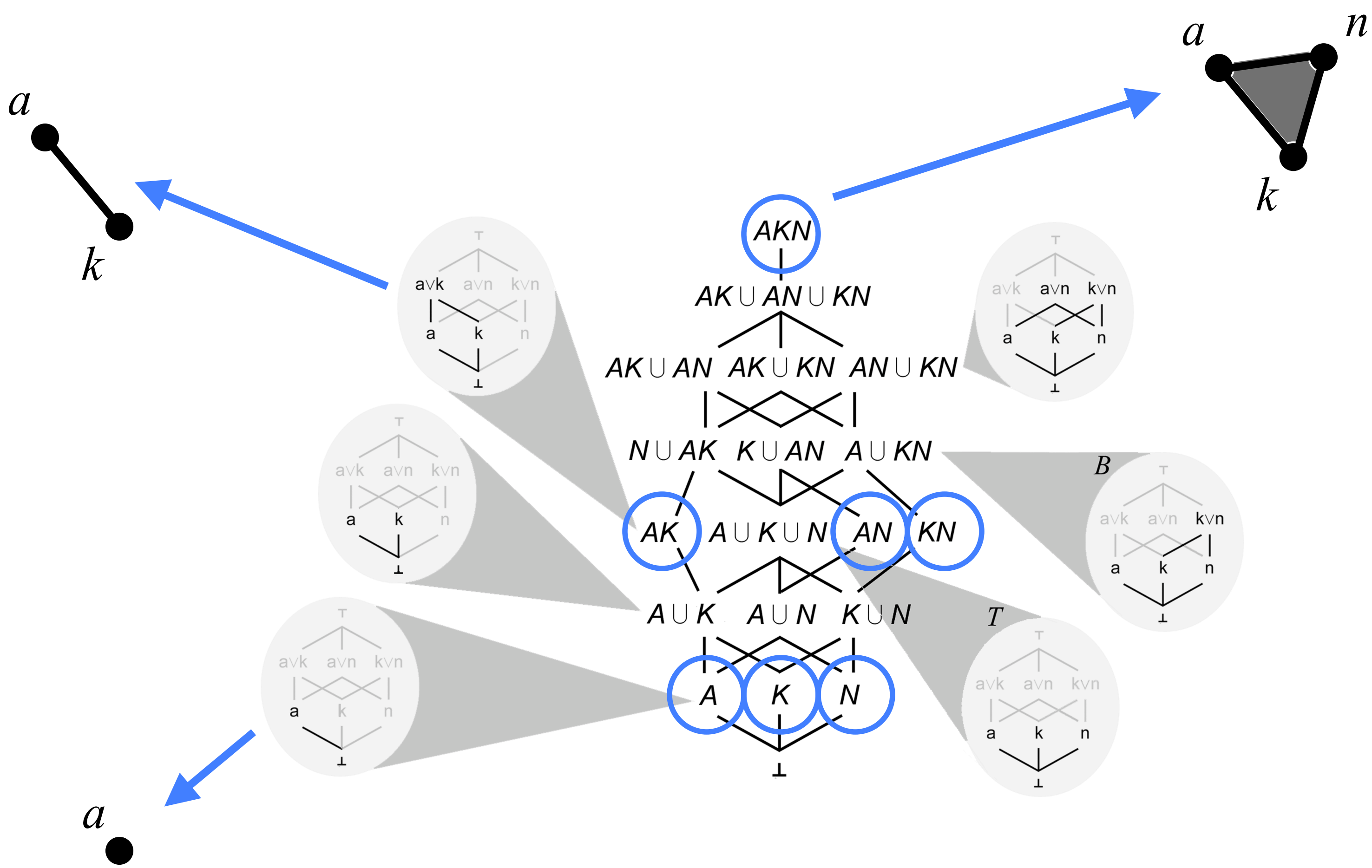


21 March 2003

Valuations on the Question Lattice



Join-Irreducible Elements



The join-irreducible elements are the simplexes.
They form a Boolean Lattice.

Valuations and Inquiry

All valuations on the question lattice can be uniquely determined from the valuations on the join-irreducible elements of the lattice.

As the lattice is associative, distributive, and commutative, there exist:

Sum Rule
Product Rule
Bayes Theorem analog

This gives us a well-defined **calculus of inquiry**!

Inductive Inquiry

Despite the earlier mistaken belief that questions followed a Boolean algebra, Ariel Caticha's derivation of the Sum and Product Rule from associativity and distributivity have allowed me to show that a consistent calculus of inquiry analogous to probability theory can indeed be constructed as imagined by Richard Cox and Robert Fry.

The valuations (probabilities) on the Boolean lattice of assertions should induce valuations (relevances) on the join-irreducible (ideal) questions, which in turn dictate the valuations on all questions.

Does entropy play a role?

Entropy

As we are free to assign valuations on the join-irreducible elements of the question lattice, we are allowed to assign entropy if we wish.

But is an entropy assignment on the question lattice consistent with the probability assignments on the assertion lattice?

Cox’s Generalized Entropy

Cox's Generalized Entropy

Cox defined the entropy of a join of two questions as

$$H(A \,\square\, B) \quad = \quad H(A,B) \,\square\, H(A) \,\square\, H(B)$$

Notice that this equation uses the inclusion-exclusion principle.

Mutual Information

Cox's generalized entropy for a rank 2 question is related to the Mutual Information

$$H(A \vee B) \quad = \quad H(A,B) - H(A) - H(B) \quad = \quad -I(A,B)$$

Mobius Functions Again

And Cox's higher-order generalization is also known as the multiinformation (McGill 1955) or co-information (Bell 2003).

They are related via the Mobius functions of the distributive lattice:

$$H(Q_{x_i}) \quad = \quad \underset{x_i \,\square\, x_j}{\square} \square(j)\, I(Q_{x_j})$$

$$I(Q_{x_i}) \quad = \quad \underset{x_i \,\square\, x_j}{\square} \square(j)\, H(Q_{x_j})$$

where $\square(j) = \square(\square 1)^{|x_j|} \quad = \begin{array}{llll} 1 & if & |x_j| & is\ odd \\ \square 1 & if & |x_j| & is\ even \end{array}$

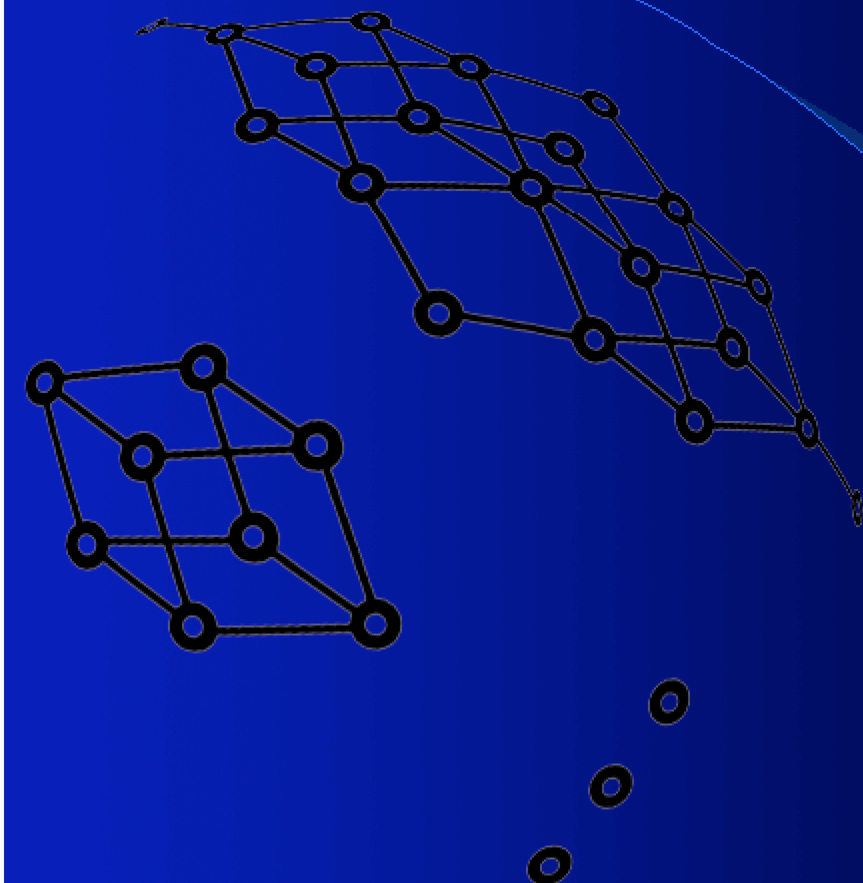
Entropies

Whatever its role, entropy must be meaningful.

I would like to thank Ariel Caticha and Carlos Rodríguez for their discussions, which have enlightened and inspired me.

I would like to thank Robert Fry for introducing me to this fascinating area of study.

Questions Revisited



Kevin H. Knuth, Ph.D.

Computational Sciences Division
NASA Ames Research Center
Moffett Field CA 94035